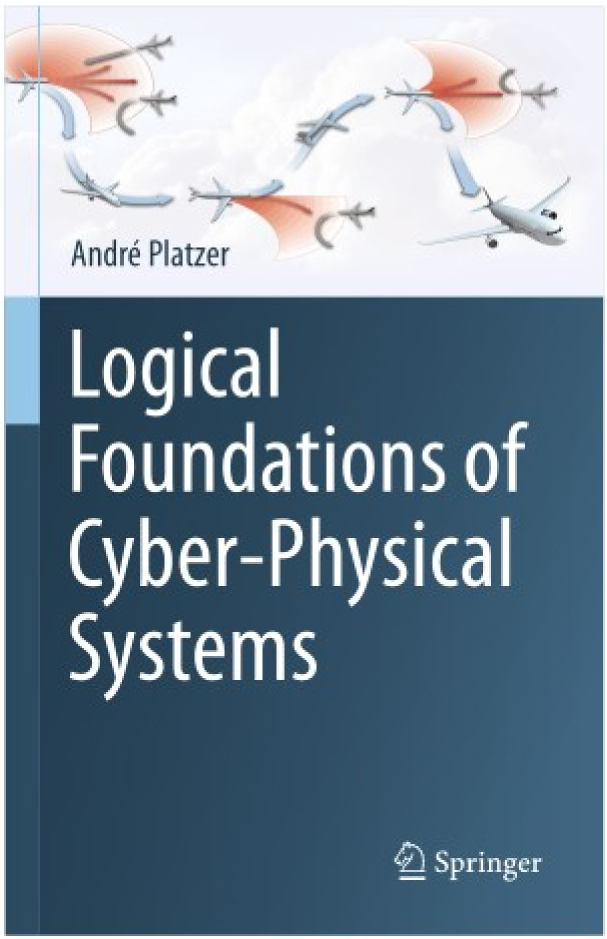
10: Differential Equations & Differential Invariants

Logical Foundations of Cyber-Physical Systems



Heavily inspired from the slides of André Platzer

SO FAR: elementary CPS (his words)

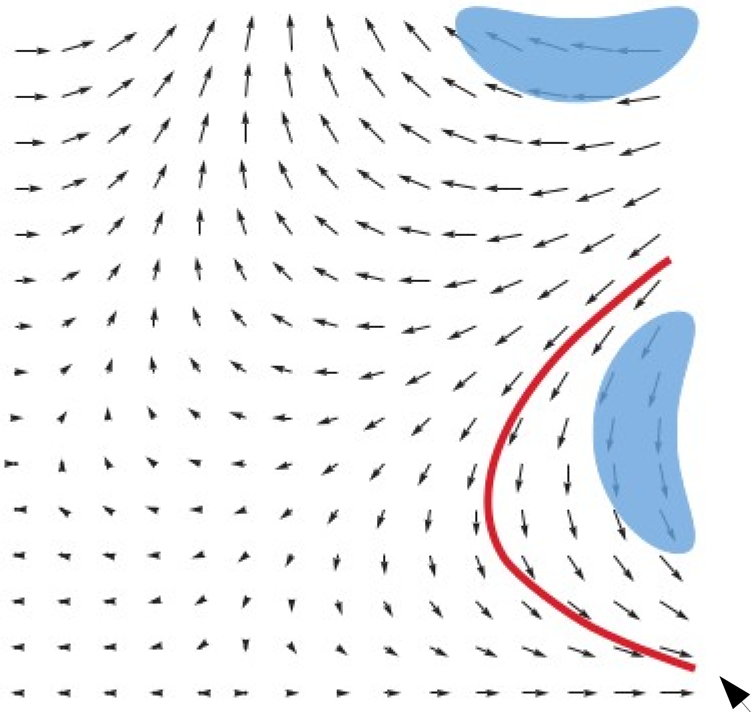
TODAY: Advance CPS

Recall from Chapter 5

The differential lemma

$$[']\ [x' = f(x)]P \leftrightarrow \forall t \geq 0\, [x := y(t)]P \qquad (y' = f(y),\ y(0) = x)$$

y captures all the behaviour that the DE could have



On saturday, Sami said
“we need to make
the math ourselves”.

Global solution for a given initial value

ODE		Solution
$x' = 1, x(0) = x_0$	Constant → linear	$x(t) = x_0 + t$
$x' = 5, x(0) = x_0$		$x(t) = x_0 + 5t$
$x' = x, x(0) = x_0$	Linear → exponential	$x(t) = x_0 e^t$
$x' = x^2, x(0) = x_0$		$x(t) = \frac{x_0}{1 - t x_0}$
$x' = \frac{1}{x}, x(0) = 1$		$x(t) = \sqrt{1 + 2t} \ldots$
$y'(x) = -2xy, y(0) = 1$		$y(x) = e^{-x^2}$
$x'(t) = tx, x(0) = x_0$		$x(t) = x_0 e^{\frac{t^2}{2}}$
$x' = \sqrt{x}, x(0) = x_0$		$x(t) = \frac{t^2}{4} \pm t\sqrt{x_0} + x_0$
$x' = y, y' = -x, x(0) = 0, y(0) = 1$		$x(t) = \sin t, y(t) = \cos t$
$x' = 1 + x^2, x(0) = 0$		$x(t) = \tan t$
$x'(t) = \frac{2}{t^3} x(t)$		$x(t) = e^{-\frac{1}{t^2}}$ non-analytic
$x' = x^2 + x^4$		???
$x'(t) = e^{t^2}$		non-elementary

=> The solutions are more complicated than the ODEs

ODE	Solution = the physical process
Local description	Global description
Simple	(More) complicated

By solving the ODES, we undo their **descriptive power**

$$\frac{d[\mathrm{Cdc13}_T]}{dt} = k_1 M - (k_2' + k_2''[\mathrm{Ste9}] + k_2'''[\mathrm{Slp1}])[\mathrm{Cdc13}_T],$$

$$\frac{d[\mathrm{preMPF}]}{dt} = k_{\mathrm{wee}}([\mathrm{Cdc13}_T] - [\mathrm{preMPF}]) - k_{25}[\mathrm{preMPF}] - (k_2' + k_2''[\mathrm{Ste9}] + k_2'''[\mathrm{Slp1}])[\mathrm{preMPF}],$$

$$\frac{d[\mathrm{Ste9}]}{dt} = (k_3' + k_3''[\mathrm{Slp1}])\frac{1-[\mathrm{Ste9}]}{J_3 + 1 - [\mathrm{Ste9}]} - (k_4'[\mathrm{SK}] + k_4[\mathrm{MPF}])\frac{[\mathrm{Ste9}]}{J_4 + [\mathrm{Ste9}]},$$

$$\frac{d[\mathrm{Slp1}_T]}{dt} = k_5' + k_5''\frac{[\mathrm{MPF}]^4}{J_5^4 + [\mathrm{MPF}]^4} - k_6[\mathrm{Slp1}_T],$$

$$\frac{d[\mathrm{Slp1}]}{dt} = k_7[\mathrm{IEP}]\frac{[\mathrm{Slp1}_T] - [\mathrm{Slp1}]}{J_7 + [\mathrm{Slp1}_T] - [\mathrm{Slp1}]} - k_8\frac{[\mathrm{Slp1}]}{J_8 + [\mathrm{Slp1}]} - k_6[\mathrm{Slp1}],$$

$$\frac{d[\mathrm{IEP}]}{dt} = k_9[\mathrm{MPF}]\frac{1-[\mathrm{IEP}]}{J_9 + 1 - [\mathrm{IEP}]} - k_{10}\frac{[\mathrm{IEP}]}{J_{10} + [\mathrm{IEP}]},$$

$$\frac{d[\mathrm{Rum1}_T]}{dt} = k_{11} - (k_{12} + k_{12}'[\mathrm{SK}] + k_{12}''[\mathrm{MPF}])[\mathrm{Rum1}_T],$$

$$\frac{d[\mathrm{SK}]}{dt} = k_{13}[\mathrm{TF}] - k_{14}[\mathrm{SK}],$$

$$\frac{dM}{dt} = \mu M,$$

$$[\mathrm{Trimer}] = \frac{2[\mathrm{Cdc13}_T][\mathrm{Rum1}_T]}{\Sigma + \sqrt{\Sigma^2 - 4[\mathrm{Cdc13}_T][\mathrm{Rum1}_T]}},$$

$$[\mathrm{MPF}] = \frac{([\mathrm{Cdc13}_T] - [\mathrm{preMPF}])([\mathrm{Cdc13}_T] - [\mathrm{Trimer}])}{[\mathrm{Cdc13}_T]},$$

$$[\mathrm{TF}] = G(k_{15}M, k_{16}' + k_{16}''[\mathrm{MPF}], J_{15}, J_{16}),$$

where

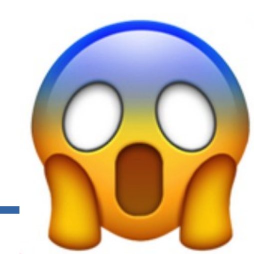
$$k_{\mathrm{wee}} = k_{\mathrm{wee}}' + (k_{\mathrm{wee}}'' - k_{\mathrm{wee}}')G(V_{\mathrm{awee}}, V_{\mathrm{iwee}}[\mathrm{MPF}], J_{\mathrm{awee}}, J_{\mathrm{iwee}}),$$

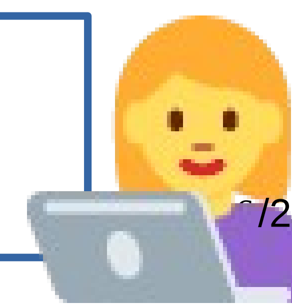
$$k_{25} = k_{25}' + (k_{25}'' - k_{25}')G(V_{a25}[\mathrm{MPF}], V_{i25}, J_{a25}, J_{i25}),$$

$$\Sigma = [\mathrm{Cdc13}_T] + [\mathrm{Rum1}_T] + K_{\mathrm{diss}},$$

$$G(a,b,c,d) = \frac{2ad}{b - a + bc + ad + \sqrt{(b - a + bc + ad)^2 - 4ad(b-a)}}.$$

Differential-algebraic sytem of the cell cycle of the fission yeast, Novak et al. 2001





$$x'' = -x \qquad\qquad x(t) = \sin(t) = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \frac{t^9}{9!} - \ldots$$

$\rightarrow$ not part of FOL arithmetics:

- the sin is not part of our synthax,
- the series need infinitely many power

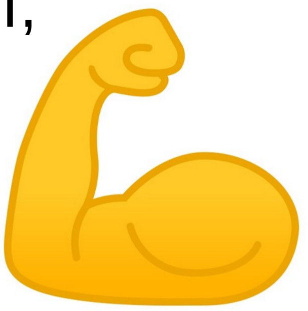
$$x''(t) = e^{t^2}$$

no elementary closed-form solution

Use the differential lemma:

$$[{}'] \quad [x' = f(x)]P \leftrightarrow \forall t \geq 0\, [x := y(t)]P \qquad (y' = f(y),\ y(0) = x)$$

Directly reasoning on the ODEs themselves
- Exploit the descriptive power of ODEs for proof,
- No need to solve ODEs anymore



Induction:
Establishing the truth of property by analysing generically the one **step** (= the **loop body**) that is executed repeatidly

Induction with discrete dynamics: ✓

Lemma 7.3 (Loop invariant rule):

$$\text{loop}\quad \frac{\Gamma \vdash J, \Delta \quad J \vdash [\alpha]J \quad J \vdash P}{\Gamma \vdash [\alpha^*]P, \Delta}$$

Induction with continous dynamics: ?

Differential invariant

$$\frac{\Gamma \vdash F, \Delta \quad F \vdash ???F \quad F \vdash P}{\Gamma \vdash [x' = f(x)]P, \Delta}$$

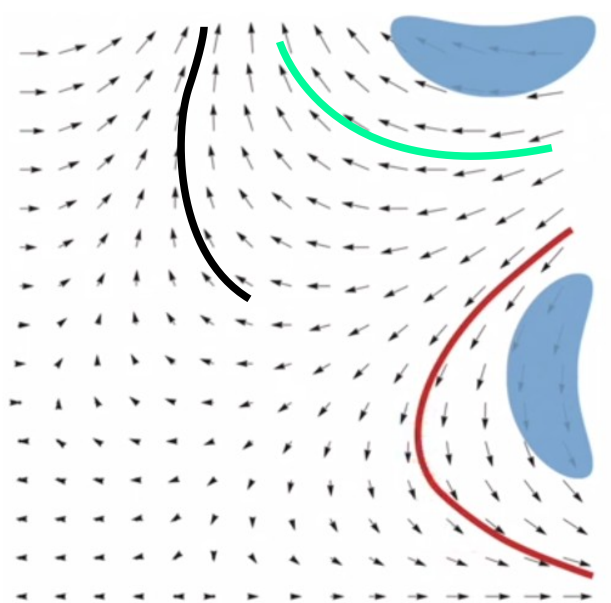
Induction with continous dynamics: ?

Differential invariant

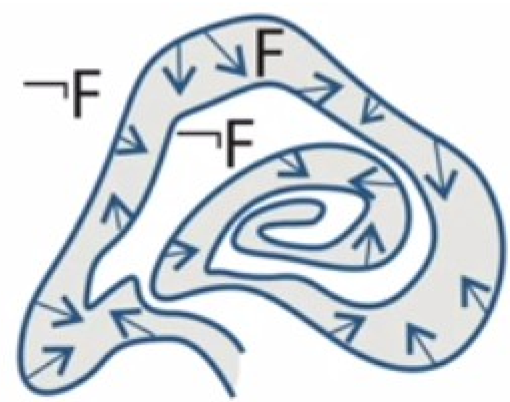
$$\frac{\Gamma \vdash F, \Delta \quad F \vdash {\color{red}???}F \quad F \vdash P}{\Gamma \vdash [x' = f(x)]P, \Delta}$$

$['] \; [x' = f(x)]P \leftrightarrow \forall t \geq 0 \, [x := y(t)]P$

$(y' = f(y),\ y(0) = x)$



Induction step with continuous time?!



→ “The system only evolves into **direction** where F”

But… we do not have logic to talk about “direction”...

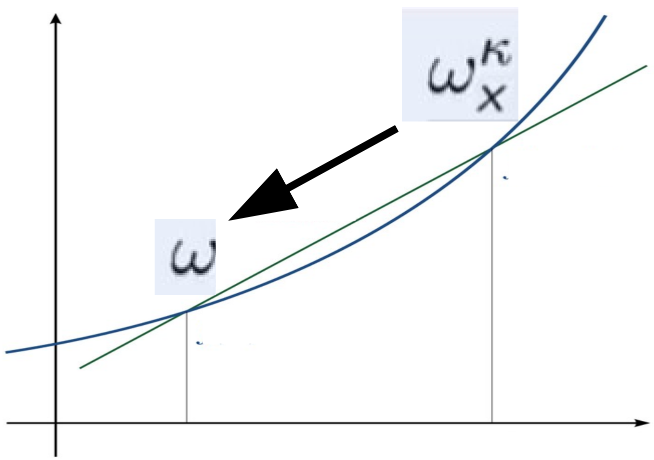
internalize primes into dL syntax → **differential** dynamic logic

$$e ::= x \mid x' \mid c \mid e + \tilde{e} \mid e - \tilde{e} \mid e \cdot \tilde{e} \mid e / \tilde{e} \mid (e)'$$

Semantics of primes:

$$\omega[\![(e)']\!] = \sum_x \omega(x') \frac{\partial [\![e]\!]}{\partial x}(\omega)$$

$$\frac{\partial [\![e]\!]}{\partial x}(\omega) = \lim_{\kappa \to \omega(x)} \frac{\omega_x^\kappa [\![e]\!] - \omega[\![e]\!]}{\kappa - \omega(x)}$$



Note that the states are enriched with x’

$$\omega[\![(e)']\!] \cancel{=} \frac{d\omega[\![e]\!]}{dt} \qquad \text{nonsense!}$$

We need something compositional
and which does not depend of time.
However...

The meaning of the syntactival expression happens to **coïncide** with the meaning of the analytic time derivative

Lemma (Differential lemma) (Differential value vs. Time-derivative)

If $\varphi \models x' = f(x) \wedge Q$ *for duration* $\boxed{r>0,}$ *then for all* $0 \le z \le r$, $FV(e) \subseteq \{x\}$:

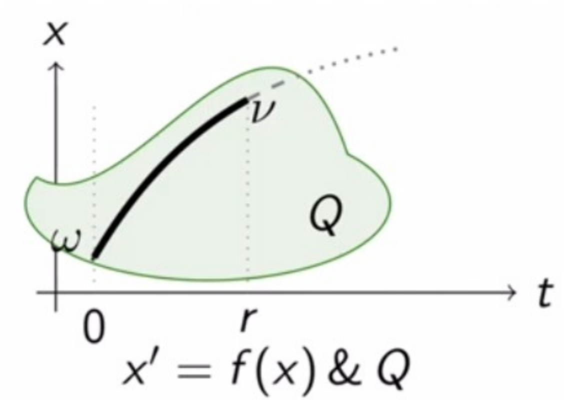
Syntactic ′ $\quad \varphi(z)[\![(e)']\!] = \frac{d\varphi(t)[\![e]\!]}{dt}(z) \quad$ Analytic ′

Good that it coïncide because we were already using it...

Definition (Hybrid program semantics) $\qquad (\llbracket\cdot\rrbracket : \mathrm{HP} \to \wp(\mathcal{S}\times\mathcal{S}))$

$$\llbracket x' = f(x) \,\&\, Q \rrbracket = \{(\varphi(0)|_{\{x'\}^{\complement}}, \varphi(r)) : \varphi(z) \models x' = f(x) \land Q \text{ for all } 0 \le z \le r$$
$$\text{for a solution } \varphi : [0, r] \to \mathcal{S} \text{ of any duration } r \in \mathbb{R}\}$$
$$\text{where } \varphi(z)(x') \stackrel{\text{def}}{=} \frac{\mathrm{d}\varphi(t)(x)}{\mathrm{d}t}(z)$$

with $\underline{\varphi(0) = \omega \text{ except on } x'}$ and $\varphi(r) = \nu$



There is an x' in all the states, but:

Initial value of x' in ω is irrelevant since defined by ODE.

Final value of x' is carried over to the final state ν.

Lemma (Differential assignment) (Effect on Differentials)

If $\varphi \models x' = f(x) \wedge Q$ *then* $\varphi \models P \leftrightarrow [x' := f(x)]P$

Logical way to expose that “while we follow the DE: x’ = f(x)”

Lemma (Derivations) (Equations of Differentials)

$$(e + k)' = (e)' + (k)'$$

$$(e \cdot k)' = (e)' \cdot k + e \cdot (k)'$$

$$(c())' = 0 \qquad \textit{for constants/numbers } c()$$

$$(x)' = x' \qquad \textit{for variables } x \in \mathcal{V}$$

$$\omega[\![(e)']\!] = \sum_x \omega(x') \frac{\partial [\![e]\!]}{\partial x}(\omega)$$

Axiomatics

(1) DE captures the differential assignement lemma (a semantic principle) to make it accessible as an axiomatic principle in the logic

Differential effect (DE)

$$[x' = f(x) \,\&\, Q]P \leftrightarrow [x' = f(x) \,\&\, Q][x' := f(x)]P$$

Lemma (Differential assignment) (Effect on Differentials)

If $\varphi \models x' = f(x) \land Q$ *then* $\varphi \models P \leftrightarrow [x' := f(x)]P$

"while we follow the DE: x' = f(x)"

Axiomatics

(2) DI captures uses the differential lemma
to make it accessible as an axiomatic principle in the logic

Differential Induction (DI)

$$([x' = f(x)]\underline{e = 0 \leftrightarrow e = 0}) \leftarrow [x' = f(x)](e)' = 0$$

0 at all time if 0 right now

No change of e
along the DE

$$\frac{\mathrm{d}\varphi(t)[\![e]\!]}{\mathrm{d}t}(z) = 0$$

Lemma (Differential lemma) (Differential value vs. Time-derivative)

If $\varphi \models x' = f(x) \wedge Q$ *for duration* $r>0$, *then for all* $0 \leq z \leq r$, $FV(e) \subseteq \{x\}$:

$$\varphi(z)[\![(e)']\!] = \frac{\mathrm{d}\varphi(t)[\![e]\!]}{\mathrm{d}t}(z)$$

Rq: it works not only for 0, but he always normalises the equations

We can pack DE and DI together in the dI proof rule:

Differential Invariant dI

$$\text{dI}\ \frac{\vdash [x' := f(x)](e)' = 0}{e = 0 \vdash [x' = f(x)]e = 0}$$

from question about DE
to question about assignment

Proof (dI is a derived rule).

$$\begin{array}{l} \text{G}\ \dfrac{\vdash [x' := f(x)](e)' = 0}{\vdash [x' = f(x)][x' := f(x)](e)' = 0} \\ \text{DE}\ \dfrac{}{\vdash [x' = f(x)](e)' = 0} \\ \text{DI}\ \dfrac{}{e = 0 \vdash [x' = f(x)]e = 0} \end{array}$$

Gödel generalisation rule, Chap. 5

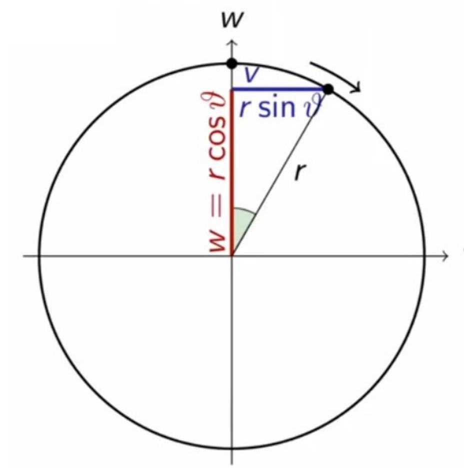
$$\text{G}\ \frac{P}{[\alpha]P}$$

$\square$

DI $([x' = f(x)]e = 0 \leftrightarrow e = 0) \leftarrow [x' = f(x)](e)' = 0$ DE $[x' = f(x)]P \leftrightarrow [x' = f(x)][x' := f(x)]P$

Example: Rotational Dynamics

$$v^2+w^2 = r^2 \rightarrow [v' = w, w' = -v]v^2+w^2 = r^2$$



$$\cfrac{\cfrac{\cfrac{\cfrac{*}{\vdash 2v(w) + 2w(-v) = 0}\mathbb{R}}{\vdash [v':=w][w':=-v]2vv' + 2ww' - 2rr' = 0}[':=]}{v^2+w^2-r^2=0 \vdash [v' = w, w' = -v]v^2 + w^2 - r^2 = 0}\mathrm{dI}}{\vdash v^2+w^2-r^2=0 \rightarrow [v' = w, w' = -v]v^2+w^2-r^2=0}\rightarrow\mathrm{R}$$

Simple proof without solving ODE, just by differentiating

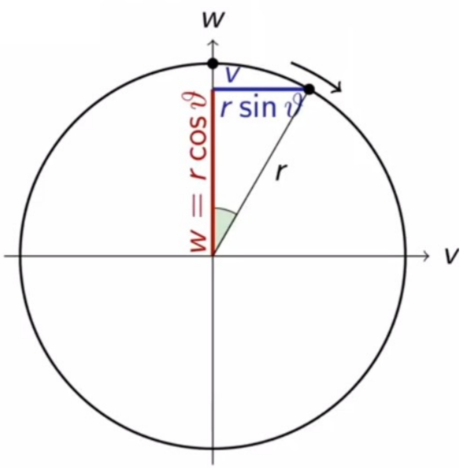
$\rightarrow$ R:
|- A $\rightarrow$ B => A |- B

Differential Invariant dI

$$\mathrm{dI}\ \frac{\vdash [x' := f(x)](e)' = 0}{e = 0 \vdash [x' = f(x)]e = 0}$$

Example: Rotational Dynamics

$$v^2+w^2=r^2 \to [v'=w,w'=-v]v^2+w^2=r^2$$



$$\to\mathrm{R}\;\dfrac{\mathrm{dI}\;\dfrac{[':=]\;\dfrac{\mathbb{R}\;\dfrac{*}{\vdash 2v(w)+2w(-v)=0}}{\vdash [v':=w][w':=-v]2vv'+2ww'-2rr'=0}}{\boxed{v^2+w^2-r^2=0} \vdash \boxed{[v'=w,w'=-v]v^2+w^2-r^2=0}}}{\vdash \boxed{v^2+w^2-r^2=0} \to \boxed{[v'=w,w'=-v]v^2+w^2-r^2=0}}$$

Simple proof without solving ODE, just by differentiating

$\to$ R:

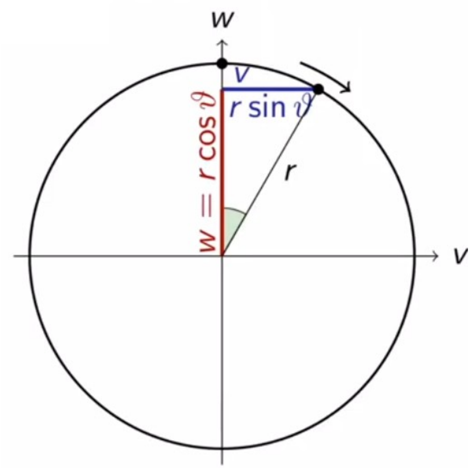
|- $\boxed{A}$ → $\boxed{B}$ => $\boxed{A}$ |- $\boxed{B}$

Differential Invariant dI

$$\mathrm{dI}\;\dfrac{\vdash [x':=f(x)](e)'=0}{e=0 \vdash [x'=f(x)]e=0}$$

Example: Rotational Dynamics

$$v^2+w^2 = r^2 \to [v' = w, w' = -v]v^2+w^2 = r^2$$



$$\dfrac{\dfrac{\dfrac{\dfrac{*}{\vdash 2v(w) + 2w(-v) = 0}\ \mathbb{R}}{\vdash [v':=w][w':=-v]2vv' + 2ww' - 2rr' = 0}\ [':=]}{\boxed{v^2+w^2-r^2=0} \vdash \boxed{[v' = w, w' = -v]}\boxed{v^2 + w^2 - r^2 = 0}}\ \mathrm{dI}}{\vdash v^2+w^2-r^2=0 \to [v' = w, w' = -v]v^2+w^2-r^2=0}\ \to\!\mathrm{R}$$

Simple proof without solving ODE, just by differentiating

$\to$ R:

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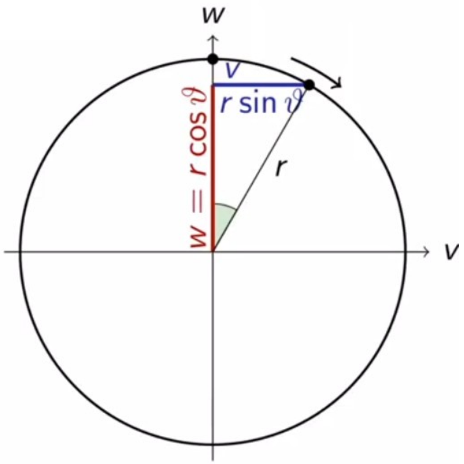
Differential Invariant dI

$$\mathrm{dI}\ \dfrac{\vdash [x' := f(x)](e)' = 0}{\boxed{e = 0} \vdash \boxed{[x' = f(x)]}\boxed{e = 0}}$$

Example: Rotational Dynamics

$$v^2+w^2 = r^2 \to [v' = w, w' = -v]v^2+w^2 = r^2$$



$$
\cfrac{\cfrac{\cfrac{\cfrac{*}{\vdash 2v(w) + 2w(-v) = 0}\ \mathbb{R}}{\vdash [v':=w][w':=-v]2vv' + 2ww' - 2rr' = 0}\ [':=]}{v^2+w^2-r^2=0 \vdash [v' = w, w' = -v]v^2 + w^2 - r^2 = 0}\ \mathrm{dI}}{\vdash v^2+w^2-r^2=0 \to [v' = w, w' = -v]v^2+w^2-r^2=0}\ \to\mathrm{R}
$$

Simple proof without solving ODE, just by differentiating

$\to$ R:
|- A $\to$ B => A |- B

Differential Invariant dI

$$\mathrm{dI}\ \frac{\vdash [x' := f(x)](e)' = 0}{e = 0 \vdash [x' = f(x)]e = 0}$$

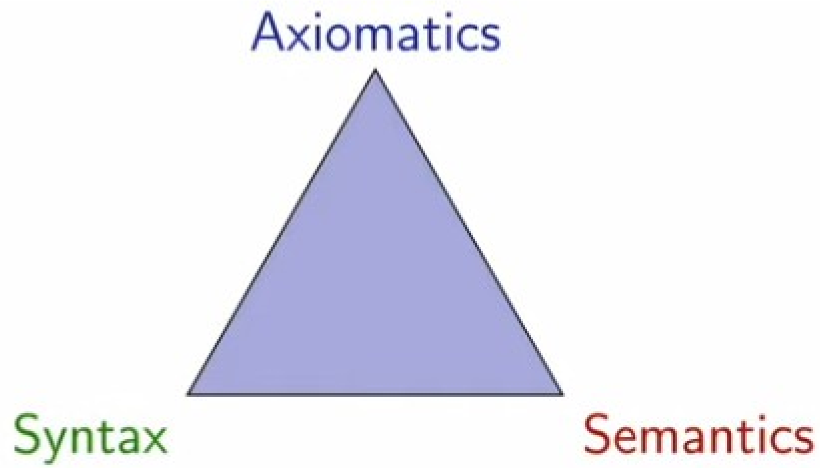
Conclusion:

Unexpected analogy between discrete and continuous dynamics
→ we found the “body loop” equivalent for continuous dynamics
→ we can now use induction without solving the ODE

say goodbye to the differential lemma,
that became superfluous

$$['] \ [x' = f(x)]P \leftrightarrow \forall t \geq 0 \, [x := y(t)]P \qquad (y' = f(y),\ y(0) = x)$$

Logical trinity:



Syntax defines the notation
What problems are we allowed to write down?

Semantics what carries meaning.
What real or mathematical objects does the syntax stand for?

Axiomatics internalizes semantic relations into universal syntactic transformations.